# Holonomy and four-dimensional manifolds

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## 1  Introduction

This paper investigates the relationship between two fundamental types of objects associated with a connection on a manifold: the existence of parallel semi-Riemannian metrics and the associated holonomy group. Typically in Riemannian geometry, a metric is specified which determines a Levi-Civita connection. Here we consider the connection as more fundamental and allow for the possibility of several parallel metrics. Holonomy is an old geometric concept which is enjoying revived interest in certain branches of mathematical physics, in particular loop quantum gravity and Calabi-Yau manifolds in string theory. It measures, in group theoretic terms the connection's deviation from flatness and takes the topology of the manifold into account.

It is well known that for a Riemannian manifold, the reducibility of the holonomy group of the Levi-Civita connection implies the existence of multiple independent parallel Riemannian metrics on the manifold

[6]. In this paper we look at the converse: when does the existence of multiple independent semi-Riemannian metrics on a manifold, parallel with respect to a linear connection, imply the reducibility of the holonomy group of the connection? We do not necessarily assume that the connection is symmetric in our solutions to this problem. If $g$ is a semi-Riemannian metric on $M$ which is parallel with respect to the connection $\nabla$, then the holonomy group $\Psi(x)$, at $x \in M$, preserves $g$: $\psi^*(g) = g$, for all $\psi \in \Psi(x)$. Thus the existence of parallel metrics places algebraic restrictions on $\Psi(x)$; these restrictions will be the subject of our investigations. For manifolds of dimension $d \neq 4$ the problem has a purely algebraic solution. For four-dimensional manifolds the relationship of the parallel metrics of the connection to the reducibility of the holonomy group is not entirely algebraic but depends also on the fundamental group of the manifold: the presence of three parallel metrics always implies reducibility but reducibiliy in the case of two metrics of signature $(2, 2)$ is guaranteed only for simply connected manifolds. The central theorem in this paper is the construction of a topologically non-trivial four-dimensional manifold with a connection that admits two independent metrics of signature $(2, 2)$ and yet has irreducible holonomy. $d = 4$ is the critical dimension with respect to reducibility of the holonomy group.

It is interesting to note that $d = 4$ appears as the critical dimension in other conttexts as well. In quantum field theory, for instance, infinite divergences appear in the calculation of scattering amplitudes as the dimension of spacetime approaches four. Also, it has been shown that $R^4$ has the remarkable property of admitting exotic differentiable structures. In superstring theory spacetime is ten or eleven dimensional but

only four dimensions are observed in nature. Therefore some unique characteristic of four-dimensional manifolds must be involved in explaining this mismatch of dimensions.

The problem of non-uniqueness of parallel metrics, largely in the case of Lorentzian connections, has been investigated by several authors (cf. [1]-[5], [7]-[8]).

## 2 Reducibility of the Holonomy Group

Let $V$ be a vector space over a field $F$, which will be either $R$ or $C$. Let $G$ be a group which acts on $V$ on the left. A subspace $W$ of $V$ is said to be *G-invariant* if $g \cdot \xi \in W$, for all $g \in G$ and $\xi \in W$. If there exists a proper, non-trivial $G$-invariant subspace of $V$ we say that $G$ *acts reducibly* on $V$, or more simply, that $G$ is *reducible*. In our applications, $G$ shall be a subgroup of $Aut(V)$, the group of linear automorphisms of $V$. The holonomy group $\Psi(x)$ of a linear connection $\nabla$ on a connected manifold $M$, at $x \in M$, is a subgroup of $Aut(T_x M)$. For any two points $x, y \in M$, the holonomy groups $\Psi(x)$ and $\Psi(y)$ are isomorphic, since $M$ is connected. If for some point (and hence all points) $x \in M$, $\Psi(x)$ is reducible (respectively, irreducible) then we say that the connection $\nabla$ has *reducible* (respectively, *irreducible*) *holonomy.* The results of this section are essentially algebraic and may be derived with the aid of standard normal form theorems in linear algrbra. Since the proofs are somewhat lengthy and technical they shall be omitted.

We begin with a theorem that provides sufficient conditions, with regard to the existence of parallel semi-Riemannian metrics, to ensure the reducibility of the holonomy group of the connection.

**Theorem 1**

(i) Let $g_1$ and $g_2$ be two independent semi-Riemannian metrics on a connected manifold $M$, parallel with respect to a linear connection $\nabla$ on $M$. Suppose that one of $g_1$, $g_2$ has signature $(p, q)$ with $p \neq q$. Then $\nabla$ has reducible holonomy.

(ii) Let $g_1$ and $g_2$ be two independent semi-Riemannian metrics on a connected, two-dimensional manifold $M$, parallel with respect to a linear connection $\nabla$ on $M$. Then $\nabla$ has reducible holonomy.

(iii) Suppose $n \not\equiv 0 \mod 4$. Let $g_1$, $g_2$ and $g_3$ be three independent semi-Riemannian metrics on a connected, $n$-dimensional manifold $M$, parallel with respect to a linear connection $\nabla$ on $M$. Then $\nabla$ has reducible holonomy.

(iv) Let $g_1$, $g_2$, $g_3$ and $g_4$ be four independent, semi-Riemannian metrics on a connected manifold $M$, parallel with respect to a linear connection $\nabla$ on $M$. Then $\nabla$ has reducible holonomy.

It is possible to construct examples of connections on manifolds that show that the numbers of parallel semi-Riemannian metrics in the above theorem are sharp. Specifically we have the following.

**Theorem 2**

(i) Let $n = 2m$ and $m \geq 3$. There exist two independent semi-Riemannian metrics on $R^n$ of signature $(m, m)$, parallel with respect to a linear connection having an irreducible holonomy group.

(ii) Let $n = 2m$, $m = 2r$ and $r \geq 2$. There exist three independent semi-Riemannian metrics on $R^n$ of signature $(m, m)$, parallel with respect to a linear connection having an irreducible holonomy group.

# 3 Four-Dimensional Manifolds

## 3.1 Simply connected four-dimensional manifolds

The following theorem is also proved algebraically and uses the fact that the holonomy group of a connection on a simply connected manifold consists of only one component (cf. [6], p 73).

**Theorem 3**

*(i) Let $M$ be a simply connected, four-dimensional manifold. Let $g_1$ and $g_2$ be two independent semi-Riemannian metrics on $M$, parallel with respect to a linear connection $\nabla$ on $M$. Then $\nabla$ has reducible holonomy.*

*(ii) Let $M$ be a connected, four-dimensional manifold. Let $g_1, g_2$ and $g_3$ be three independent semi-Riemannian metrics on $M$, parallel with respect to a linear connection $\nabla$ on $M$. Then $\nabla$ has reducible holonomy.*

## 3.2 Irreducibility of the holonomy group

In this section we shall construct an example of a (non-simply connected) four-dimensional manifold endowed with two independent semi-Riemannian metrics of signature $(2, 2)$ which are parallel with respect to a linear connection having an irreducible holonomy group.

Let $\mathcal{F}$ be the subgroup of $GL(4; R)$ whose elements $F$ are of the form

$$F = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

The map $\phi : \mathcal{F} \to GL(2; C)$ defined by

$$\phi(F) := A + iB$$

is a group monomorphism. Let $\mathcal{G}$ be any subgroup of $\mathcal{F}$. We denote by $\mathcal{G}_C$ the subgroup $\phi(\mathcal{G})$ of $GL(2; C)$. $\phi : \mathcal{G} \to \mathcal{G}_C$ is a Lie group isomorphism. Let $C_R^2$ denote the vector space of 2-tuples of complex numbers over the real field. The following lemma is easily established.

**Lemma 4** $\mathcal{G}$ *acts reducibly on* $R^4$ *if and only if* $\mathcal{G}_C$ *acts reducibly on* $C_R^2$.

Let $K$ and $L$ denote the matrices in $GL(4; R)$ defined as follows

$$K := \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix} \quad and \quad L := \begin{pmatrix} 0 & I_{2\times2} \\ I_{2\times2} & 0 \end{pmatrix}.$$

$K$ and $L$ may be regarded as symmetric bilinear forms on $R^n$ with signature $(2, 2)$. Let $\mathcal{H}$ be the subgroup of $GL(4; R)$ whose elements $H$ satisfy

$$\begin{cases} {}^t HKH &= K, \ and \\ {}^t HLH &= L. \end{cases}$$

Let $G \in \mathcal{F}$ be defined by

$$G := \left( \begin{array}{cc|cc} \sqrt{2} & 0 & 0 & 1 \\ 0 & -\sqrt{2} & 1 & 0 \\ \hline 0 & -1 & \sqrt{2} & 0 \\ -1 & 0 & 0 & -\sqrt{2} \end{array} \right),$$

Then $G \in \mathcal{H}$ and $G^2 = I$. Let $\mathcal{G}$ denote the subgroup of $\mathcal{F}$ consisting of the two elements $I$ and $G$: $\mathcal{G} := \{I, G\}$. The characteristic polynomial $p$ of $G$ is given by $p(x) = (x^2 - 1)^2$. The eigenspace $E$ of $G$ corresponding

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to the eigenvalue $\lambda = 1$ is a two-dimensional subspace of $R^4$. Set $M' := R^4 - E$. $\mathcal{G}$ acts properly discontinously (cf. [6] Vol I, pp 43-44) on the left of $M'$, by matrix multiplication. Let $g_1$ and $g_2$ denote the semi-Riemannian metrics on $M'$ represented, respectively, by $K$ and $L$ in the moving frame $(dx^1, ..., dx^4)$ on $M'$. We see that $L_G^*(g_i) = g_i$, for $i = 1, 2$, where $L_G$ denotes left multiplication by $G$.

Define the 1-forms $\alpha$ and $\beta$ on $M'$ by

$$\alpha := x^1 dx^4 - x^4 dx^1 \qquad and \qquad \beta := x^2 dx^3 - x^3 dx^2$$

and the matrix of 1-forms, $\theta$, by

$$\theta := \left( \begin{array}{cc|cc} 0 & \alpha & 0 & \beta \\ -\alpha & 0 & -\beta & 0 \\ \hline 0 & -\beta & 0 & \alpha \\ \beta & 0 & -\alpha & 0 \end{array} \right).$$

The following two lemmas may be demonstrated by direct calculation.

**Lemma 5** $G\theta = -\theta G$.

**Lemma 6** $L_G^*(\theta) = -\theta$.

Define the linear connection $\nabla$ on $M'$ by

$$\nabla_{X_i} dx^j := - \sum_{l=1}^{4} \theta_l^j(X_i) dx^l,$$

for $1 \leq i, j \leq 4$, where $X_i = \partial/\partial x^i$.

**Lemma 7** $g_1$ and $g_2$ are parallel with respect to $\nabla$.

**Proof:** A metric $g = \sum_{i,j=1}^{4} C_{ij} dx^i \otimes dx^j$ on $M'$, with $C \in GL(4; R)$, is parallel with respect to $\nabla$ if and only if ${}^t\theta C + C\theta = 0$. The lemma follows from the fact that ${}^t\theta K + K\theta = 0$ and ${}^t\theta L + L\theta = 0$.

**Q.E.D.**

For a vector field $X$ on $M'$, we denote $L_G(X) := L_{G*} \circ X \circ L_G$.

**Lemma 8** *Let $X, Y$ be vector fields on $M'$. Then*

$$L_G(\nabla_X Y) = \nabla_{L_G(X)} L_G(Y).$$

**Proof:** It suffices to consider $X = X_i = \partial/\partial x^i$ and $Y = X_j = \partial/\partial x^j$.

$$
\begin{aligned}
L_G(\nabla_{X_i} X_j)(x) &= \sum_{s=1}^{4} \sum_{t=1}^{4} G_s^t \theta_j^s (X_i|_{Gx}) X_t(x) \\
&= -\sum_{s=1}^{4} \sum_{t=1}^{4} G_j^s \theta_s^t (X_i|_{Gx}) X_t(x), & by \ Lemma \ 5 \\
&= \sum_{s=1}^{4} \sum_{t=1}^{4} G_j^s L_G^*(\theta_s^t)(X_i|_{Gx}) X_t(x), & by \ Lemma \ 6 \\
&= \nabla_{L_G(X_i)} L_G(X_j)(x).
\end{aligned}
$$

**Q.E.D.**

In summary, the left group action of $\mathcal{G}$ on $M'$ preserves $g_1$, $g_2$ and $\nabla$. It follows that $g_1$, $g_2$ and $\nabla$ project to the quotient manifold

$$M := M'/\mathcal{G}.$$

We denote the projections of $g_1$, $g_2$ and $\nabla$ to $M$ by the same symbols.

Now let $\gamma : [0, 1] \to M'$ be the smooth curve defined by

$$\gamma(t) := \begin{pmatrix} 1 - 2t \\ t(t-1) \\ t(t-1) \\ (\sqrt{2}+1)(2t-1) \end{pmatrix}.$$

Set $x_0 := \gamma(0)$ and $x_1 := \gamma(1)$. Since $x_1 = Gx_0$, $\gamma$ is a closed loop in $M$. Let $\tau' : T_{x_0}M' \to T_{x_1}M'$ denote parallel translation along $\gamma$ in $M'$ and let $\tau : T_{x_0}M \to T_{x_0}M$ denote parallel translation along $\gamma$ in $M$. Let $a^j X_j|_{x_0} \in T_{x_0}M'$, where we have used the summation convention. Then $\tau'(a^j X_j|_{x_0}) = f^j(1)X_j|_{x_1}$, where $f : [0, 1] \to R^4$ is the unique curve satisfying

$$\begin{cases} \dot{f} + \theta(\dot{\gamma})f = 0, & and \\ f^j(0) = a^j, & 1 \le j \le 4. \end{cases}$$

Now $\theta(\dot{\gamma}) \equiv 0$ and so $f = f(t)$ is constant. Therefore $\tau'(a^j X_j|_{x_0}) = a^j X_j|_{x_1}$. It follows that

$$\tau(a^j X_j|_{x_0}) = G^j_k a^k X_j|_{x_0}.$$

We identify $T_{x_0}M$ with $R^4$ by means of the basis $(X_1, ..., X_4)$ and hence the holonomy group $\Psi(x_0)$ of $\nabla$ at $x_0 \in M$ is identified with elements of $GL(4; R)$. We have shown above that $G \in \Psi(x_0)$.

The curvature form $\Omega$ of $\nabla$ is

$$\Omega = d\theta + \theta \wedge \theta = d\theta = \left( \begin{array}{cc|cc} 0 & d\alpha & 0 & d\beta \\ -d\alpha & 0 & -d\beta & 0 \\ \hline 0 & -d\beta & 0 & d\alpha \\ d\beta & 0 & -d\alpha & 0 \end{array} \right).$$

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Now $d\alpha = 2dx^1 \wedge dx^4$ and $d\beta = 2dx^2 \wedge dx^3$. The Lie algebra $\psi^{inf}(x_0)$ of the infinitesimal holonomy group $\Psi^{inf}(x_0)$ of $\nabla$ at $x_0 \in M$ consists of elements $S \in \mathcal{F}$ of the form

$$
S := \left(
\begin{array}{cc|cc}
0 & a & 0 & b \\
-a & 0 & -b & 0 \\
\hline
0 & -b & 0 & a \\
b & 0 & -a & 0
\end{array}
\right),
$$

where $a, b \in R$. The Lie algebra $\psi_C^{inf}(x_0)$ of the complexified infinitesimal holonomy group $\Psi_C^{inf}(x_0)$ consists of elements $s \in GL(2; C)$ of the form

$$
s = \left(
\begin{array}{cc}
0 & a \\
-a & 0
\end{array}
\right),
$$

where $a \in C$. That is, $\psi_C^{inf}(x_0) = so(2; C)$ and so $\Psi_C^{inf}(x_0) = SO(2; C)$.

Let $\mathcal{A}$ denote the $R-$subalgebra of $GL(2; C)$ generated by $g$ and $SO(2; C)$ over $R$. It is not difficult to demonstrate the following lemma.

**Lemma 9** $\mathcal{A} = gl(2; C)$.

Let $\mathcal{H}$ be the subgroup of $GL(4; R)$ such that $\mathcal{H}_C$ is the subgroup of $GL(2; C)$ generated by $g$ and $SO(2; C)$. By Lemma 9, $\mathcal{H}_C$ acts irreducibly on $C_R^2$. By Lemma 4, $\mathcal{H}$ acts irreducibly on $R^4$. Since $\mathcal{H} \subseteq \Psi(x_0)$, $\Psi(x_0)$ acts irreducibly on $T_{x_0}M$. We arrive at the following theorem.

**Theorem 10** *$g_1$ and $g_2$ are two independent semi-Riemannian metrics of signature $(2, 2)$ on a non-simply connected, four-dimensional manifold $M$, parallel with respect to a linear connection $\nabla$ on $M$. Moreover, $\nabla$ has irreducible holonomy.*

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